ISRAEL JOURNAL OF MATHEMATICS, Vol. 74, Nos. 2-3, 1991

MULTI-DIMENSIONALITY

BY

SAHARON SHELAH†

The Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel; and Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

ABSTRACT

We prove that if T is stable, not multi-dimensional theory, then there is an infinite indiscernible set orthogonal to the empty set. This completes the proof that if $\aleph_\alpha = \aleph_\alpha^{|T|} > \aleph_\beta \geq \kappa_r(T)$, then T has $\geq 2^{|\alpha-\beta|}$ non-isomorphic $\aleph_\beta$-saturated models of cardinality $\aleph_\alpha$.

§0. Introduction

In [Sh-a, §5] we have dealt with the dividing line "T stable not multi-dimensional" for quite saturated models. The point is that as we are not assuming superstability, we do not know regular types exist, so dealing with dimensions is harder. One side of the dichotomy [Sh-a, V5.9] states that if T is stable multi-dimensional $\kappa_r(T) \leq \aleph_\alpha < \aleph_\beta$, T stable in $\aleph_\beta$, *then* T has $\geq 2^{|\beta-\alpha|}$ non-isomorphic, $\aleph_\alpha$-saturated models of power $\aleph_\beta$. In the proof we essentially use an $\mathbf{F}^a_{\aleph_\alpha}$-prime model M_S over $\bigcup_{\lambda\in S}\mathbf{I}_\lambda$, where $S \subseteq \{\aleph_\gamma : \alpha \leq \gamma \leq \beta\}$ (and $\aleph_\beta \in S$), $\mathbf{I}_\lambda$ is indiscernible over $A \cup \bigcup\{\mathbf{I}_\mu : \mu \in S\setminus\{\lambda\}\}$, $|\mathbf{I}_\lambda| = \lambda$, for every $\bar{a}_1^\lambda, \bar{a}_2^\lambda, \ldots \in \mathbf{I}_\lambda$, $\mathrm{stp}(\langle \bar{a}_1^\lambda, \bar{a}_2^\lambda, \ldots, \rangle, A)$ does not depend on λ, and claim $\{\dim(\mathbf{I}, M_S) : \mathbf{I} \subseteq M \text{ indiscernible}\}$ is S.

However, E. Hrushovski and E. Bouscaren note that a point addressed in the middle of the proof is ignored in the end: if $|S| > \lambda$, maybe $\dim(\mathbf{I}_\lambda, M_S) > \lambda$.

This is corrected here by giving a better equivalent form to a stable theory being multi-dimensional: there is an infinite indiscernible set $\mathbf{I}$ with $\mathrm{Av}(\mathbf{I},\mathbf{I})$ orthogonal to $\emptyset$.

So the proof of [Sh-a, V5.7] works. I thank Udi Hrushovski for discussion on this problem. The references to [Sh-a] can be replaced by [Sh-b].

†Originally written November 5, 1988. Publication 429.

Partially supported by the Israel–United States Binational Science Foundation; I thank Alice Leonhardt for the beautiful typing.

Received November 16, 1989 and in revised form December 19, 1990.

NOTATION. $\mathrm{Cb}(p)$ denotes the canonical base of the type p (see [Sh-a, III§6]); $\mathrm{ctp}(p)$, canonical type (essentially $p \restriction \mathrm{Cb}(p)$) — see there. $\perp$ denotes orthogonal; $\perp_{\mathrm{W}}$, weakly orthogonal.

$B \underset{A}{\bigcup} C$ means $\{B,C\}$ is independent over A; $B \underset{A}{\not\bigcup} C$, the negation of $\bigcup$.

$p \restriction_{\mathrm{st}} B$ is the unique $q \in S^{<\omega}(B)$ parallel to p (if there is one and only one such q).

$\leq_{\mathrm{w}}$ (see [Sh-a, V§5]), i.e. $\{p_i : i < i^*\} \leq_{\mathrm{w}} q$, *if* for every $\lambda > |\bigcup_i \mathrm{Dom}\, p_i \cup \mathrm{Dom}\, q| + \kappa_r(T)$ and $\mathbf{F}^a_\lambda$-saturated model M including $\bigcup_i \mathrm{dom}\, p_i \cup \mathrm{Dom}\, q$, we have $\dim(q,M) \geq \min\{\dim(p_i,M) : i < i^*\}$.

$\dfrac{\bar a}{B}$ the type of $\bar a$ over B.

§1. Sharpening the multi-dimensionality dividing line (for stable theories)

HYPOTHESIS. *T Stable, $\kappa(T) > \aleph_0$.*

1.1. CLAIM. Suppose

(a) $\kappa = \kappa_r(T) + \aleph_1$;

(b) $M_0 < M_1 < M_2$, $\|M_1\| = \lambda$;

(c) for every $\bar a \in {}^{\omega>}(M_2)$, if $\bar a \notin {}^{\omega>}M_1$ then $\dim\left(\mathrm{ctp}\left(\dfrac{\bar a}{M_1}\right), M_2\right) > \lambda$;

(d) $\mathbf{J} = \{c_\zeta : \zeta \leq \kappa\} \subseteq M_2$ is indiscernible, $\mathrm{Av}(\mathbf{J},\mathbf{J}) \perp p$ for every $p \in S(M_0)$ satisfying $\dim(p,M_2) > \lambda$; also c_κ realizes $\mathrm{Av}(\mathbf{J}, M_1 \cup \{c_\zeta : \zeta < \kappa\})$;

(e) M_0, M_1 are $\mathbf{F}^a_\kappa$-saturated;

(f) *if* $p_i \in S(M_0)$ for $i < \kappa$, $A \subseteq M_0$, $|A| < \kappa$, $B \subseteq M_2$, $q \in S(B)$ stationary, $|B| < \kappa$, $\bigwedge_{i<\kappa} q \not\perp p_i$ and $\dim(q,M_2) > \lambda$, *then* there are $q' \in S(B')$, $B' \subseteq M_0$, $\bigwedge_i q' \not\perp p_i$, $\dim(q',M_2) > \lambda$, and B', B realize the same type over A;

(g) if $\bar c \in {}^{\omega}M_2$, $\bar c \notin {}^{\omega}M_0$, then $\dim\left(\mathrm{ctp}\left(\dfrac{\bar c}{M_0}\right), M_0\right) > \kappa$.

Then $\mathrm{Av}(\mathbf{J},\mathbf{J}) \perp M_0$.

PROOF. Assume not. Let $\mathbf{I} = \{c_n : n < \omega\}$; there is $A \subseteq M_0$ such that $|A| < \kappa$, $\dfrac{\langle c_n : n < \omega\rangle}{M_0}$ does not fork over A. By assumption (g) we can find, for $\alpha < \kappa^+$, $\langle c^\alpha_n : n < \omega\rangle \in M_0$ such that $\{\langle c^\alpha_n : n < \omega\rangle : \alpha < \kappa^+\}$ is independent over A, each realizing $\mathrm{stp}(\langle c_n : n < \omega\rangle, A)$. Let $\mathbf{I}^\alpha = \{c^\alpha_n : n < \omega\}$; by [Sh-a, V3.4] (the assumption is the conclusion fails) $\mathrm{Av}(\mathbf{J},\mathbf{J})$ is not orthogonal to A and not orthogonal to $\mathrm{Av}(\mathbf{I}^\alpha,\mathbf{I}^\alpha)$ for $\alpha < \kappa^+$. For each $\alpha < \kappa^+$, $\mathrm{Av}(\mathbf{I}^\alpha,\mathbf{I}^\alpha)$ cannot be orthogonal to $\dfrac{\langle c_\zeta : \zeta \leq \kappa\rangle}{M_1}$ [as then let M_1' be $\mathbf{F}^a_\kappa$-primary over $M_1 \cup \{c_\alpha : \alpha < \kappa\}$, M_1'' $\mathbf{F}^a_\kappa$-prime over $M_1' \cup \{c_\kappa\}$; now by [Sh-a, V4.10(2)]

$$\frac{c_\kappa}{M_1 \cup \{c_\zeta : \zeta < \kappa\}} \vdash \frac{c_\kappa}{M_1'}$$

hence M_1'' is $\mathbf{F}^a_\kappa$-primary over $M_1 \cup \{c_\alpha : \alpha \leq \kappa\}$;

$$\text{if } \mathrm{Av}(\mathbf{I}^\alpha,\mathbf{I}^\alpha) \perp \frac{\langle c_\zeta : \zeta \leq \kappa\rangle}{M_1}, \quad \text{then } \mathrm{Av}(\mathbf{I}^\alpha, M_1) \perp_{\mathrm{w}} \frac{\langle c_\alpha : \alpha \leq \kappa\rangle}{M_1}$$

hence $\mathrm{Av}(\mathbf{I}^\alpha, M_1) \vdash \mathrm{Av}(\mathbf{I}^\alpha, M_1'')$, hence by monotonicity $\mathrm{Av}(\mathbf{I}^\alpha, M_1') \vdash \mathrm{Av}(\mathbf{I}^\alpha, M_1' \cup c_\kappa)$ hence, by [Sh-a, V1.2(3)], we have

$$\mathrm{Av}(\mathbf{I}^\alpha, M_1') \perp \frac{c_\kappa}{M_1'} = \mathrm{Av}(\mathbf{I}, M_1'),$$

a contradiction].

So for some finite $u \subseteq \kappa + 1$,

$$\frac{\langle c_\zeta : \zeta \in u\rangle}{M_1} \not\perp \mathrm{Av}(\mathbf{I}^\alpha,\mathbf{I}^\alpha).$$

Without loss of generality u does not depend on α and necessarily (see [Sh-a, V1.1(1)]) $\langle c_\zeta : \zeta \in u\rangle \in M_2$ but $\langle c_\zeta : \zeta \in u\rangle \notin M_1$. By assumption (c) $\dim(\langle c_\zeta : \zeta \in u\rangle / M_1, M_2) > \lambda$, hence by assumption (f) we can find $q \in S(M_0)$ such that, for $\alpha < \kappa$, $q \not\perp \mathrm{Av}(\mathbf{I}^\alpha,\mathbf{I}^\alpha)$ and $\dim(q, M_2) > \lambda$. Without loss of generality $\{\langle c^\alpha_n : n < \omega\rangle : \alpha < \kappa\}$ is independent over $(A \cup \mathrm{Cb}(q), A)$. As also $\{\langle c^\alpha_n : n < \omega\rangle : \alpha < \kappa\} \cup \{\langle c_n : n < \omega\rangle\}$ is necessarily independent over $(A \cup \mathrm{Cb}(q), A)$, q is also not orthogonal to $\mathrm{Av}(\mathbf{I},\mathbf{I})$, but this contradicts assumption (d).

1.2. Claim. (1) If $B \subseteq M_1$, $\mathcal{P} \subseteq S(B)$, $\|M_1\| = \lambda$ $(> |T| + |B|)$, $\kappa = \kappa_r(T)$, M_1 is $\mathbf{F}^a_\kappa$-saturated, $\kappa > |B|$ and, for each $p \in \mathcal{P}$, $\dim(p, M_1) = \lambda$, *then* we can find $\mathbf{I}$ such that:

$(*)^{\mathbf{I}}_{B,M_1,\mathcal{P}}$ $\mathbf{I} \subseteq \bigcup_{p\in\mathcal{P}} p(M_1)$, $\mathbf{I}$ independent over B, and for each $p \in \mathcal{P}$, letting p^+ be the stationarization of p over M_1,

$$p^+ \restriction (B \cup \mathbf{I}) \vdash p^+.$$

(2) If $\mathbf{J} \subseteq M_1$ is independent over B, $|\mathbf{J}| < \lambda$ we can demand $\mathbf{J} \subseteq \mathbf{I}$, $\mathbf{I} \setminus \mathbf{J} \subseteq \bigcup_{p\in\mathcal{P}} p(M_1)$.

PROOF. (1) Without loss of generality $\mathcal{P}$ is a non-empty set of non-algebraic types and we work in $\mathfrak{C}^{eq}$. Let $\{\bar{c}_\alpha : \alpha < \lambda\}$ list all finite sequences from M_1. We choose by induction on $\alpha < \lambda$, $\bar{b}_\alpha \in {}^{\omega>}M_1$ such that:

(i) $\bar{b}_\alpha \in \bigcup_{p \in \mathcal{P}} p(M_1)$;

(ii) $\{\bar{b}_\beta : \beta \leq \alpha\}$ is independent over B;

(iii) for each α, let $\gamma(\alpha)$ be the minimal $\gamma < \lambda$ such that for some $p \in \mathcal{P}$, $p^+ \restriction_{st} (B \cup \{\bar{b}_\beta : \beta < \alpha\})$ is not weakly orthogonal to $\dfrac{\bar{c}_\gamma}{B \cup \{\bar{b}_\beta : \beta < \alpha\}}$, and then $\dfrac{\bar{b}_\alpha}{B \cup \{\bar{b}_\beta : \beta < \alpha\} \cup \bar{c}_{\gamma(\alpha)}}$ fork over $B \cup \{\bar{b}_\beta : \beta < \alpha\}$ (equivalently over B), or, if this is impossible, $\dfrac{\bar{c}_{\gamma(\alpha)}}{B \cup \{\bar{b}_\beta : \beta < \alpha\}}$ has at least two extensions which are complete types over $B \cup \{\bar{b}_\beta : \beta \leq \alpha\}$ and does not fork over $B \cup \{\bar{b}_\beta : \beta < \alpha\}$.

Easily this suffices (note: $|\{\alpha : \gamma(\alpha) = \gamma\}| < |T|^+$). The least trivial part is that given $\alpha, \gamma(\alpha)$ we can find $\bar{b}_\alpha$ satisfying (i), (ii), (iii).

By the choice $\gamma(\alpha)$, the non-trivial case is that there is $\bar{b}'_\alpha \in {}^{\omega>}\mathfrak{C}$ such that

$$\bar{b}'_\alpha \underset{B \cup \{\bar{b}_\beta : \beta < \alpha\}}{\uplus} \bar{c}_{\gamma(\alpha)} \quad \text{and} \quad \frac{\bar{b}'_\alpha}{B \cup \{\bar{b}_\beta : \beta < \alpha\}}$$

is a stationarization of some $p_\alpha \in \mathcal{P}$. Now choose, by induction on ζ, $\bar{b}_{\alpha,\zeta} \in {}^{\omega>}\mathfrak{C}$ such that:

$$\bar{b}_{\alpha,\zeta} \underset{B}{\bigcup} \{\bar{b}_\beta : \beta < \alpha\} \cup \bar{c}_{\gamma(\alpha)} \cup \{\bar{b}_{\alpha,\xi} : \xi < \zeta\}$$

and

$$\frac{\bar{b}_{\alpha,\zeta}}{B} \in \mathcal{P} \quad \text{and} \quad \bar{b}'_\alpha \underset{B \cup \{\bar{b}_\beta : \beta < \alpha\} \cup \bar{c}_{\gamma(\alpha)} \cup \{\bar{b}_{\alpha,\xi} : \xi < \zeta\}}{\uplus} \bar{b}_{\alpha,\zeta}.$$

For some $\zeta < \kappa(T)$, $\bar{b}_{\alpha,\xi}$ is defined iff $\xi < \zeta$. We can also find $u \subseteq \alpha$, $|u| < \kappa$ such that

$$\bar{b}'_\alpha \cup \bar{c}_{\gamma(\alpha)} \cup \bigcup_{\xi < \zeta} \bar{b}_{\alpha,\xi} \underset{B \cup \{\bar{b}_\beta : \beta \in u\}}{\bigcup} B \cup \{\bar{b}_\beta : \beta \in \alpha \setminus u\}.$$

Easily

$$\otimes \qquad \mathrm{stp}\left(\frac{\bar{b}'_\alpha}{B \cup \{\bar{b}_\beta : \beta \in u\} \cup \bar{c}_{\gamma(\alpha)} \cup \{\bar{b}_{\alpha,\xi} : \xi < \zeta\}}\right) \vdash \mathrm{stp}\left(\frac{\bar{b}'_\alpha}{B \cup \{\bar{b}_\beta : \beta < \alpha\} \cup \bar{c}_{\gamma(\alpha)} \cup \{\bar{b}_{\alpha,\xi} : \xi < \zeta\}}\right).$$

Now as $\dim(p,M_1)=\lambda$ for $p\in\mathcal{P}$, there is an elementary mapping f such that $f\restriction(B\cup\{\bar b_\beta:\beta<\alpha\}\cup\bar c_{\gamma(\alpha)})=\mathrm{id}$ and, for $\xi<\zeta, f(\bar b_{\alpha,\xi})\subseteq M_1$. As M_1 is $\mathbf{F}^a_\kappa$-saturated, by $\otimes$ above without loss of generality $f(\bar b'_\alpha)\subseteq {}^{\omega>}M_1$.

Let $\bar b_\alpha=f(\bar b'_\alpha)$.

(2) Same proof.

1.3. Claim. If $\lambda=\lambda^{<\kappa}>\kappa^+$, T stable in κ, $\mathrm{cf}\,\kappa>\kappa_r(T)+\aleph_1$, $\kappa\ge|T|$, M_2 is λ-saturated of power λ^+, $A^*\subseteq B^*\subseteq M_2$, $\|A\|\le\kappa$, *then* there are M_0, $M_0<M_2$, $A^*\subseteq M_0$, $\|M_0\|=\kappa$ and $\mathbf{I}\subseteq M_2$ independent over M_0 such that for each $p\in S(M_0)$, $p^+=: p\restriction_{\mathrm{st}}M_2$ (the stationarization of p over M_2) satisfies $p^+\restriction(M_0\cup\mathbf{I})\vdash p^+$ and $\mathrm{tp}(M_0,B)$ does not fork over $M_0\cap B^*$.

Proof. We choose, by induction on $\alpha<\kappa$, $M_{0,\alpha}$,$M_{1,\alpha}$ such that:

(a) $M_{0,\alpha}<M_{1,\alpha}<M_2$,

(b) $\|M_{0,\alpha}\|=\kappa$, $\|M_{1,\alpha}\|=\lambda$,

(c) $M_{0,\alpha}$ is saturated increasing in α,

(d) $M_{1,\alpha}$ is saturated increasing in α,

(e) *if* $c\in M_2$ (or $\bar c\in{}^{\omega>}M_2$),

$$\dim\left(\frac{c}{M_{1,\alpha}}\restriction\mathrm{Cb}\left(\frac{c}{M_{1,\alpha}}\right),M_2\right)\le\lambda,$$

then there is a maximal $\mathbf{I}\subseteq M_2$ independent over $\mathrm{Cb}\left(\dfrac{c}{M_{1,\alpha}}\right)$ of elements realizing $c/Cb\left(\dfrac{c}{M_{1,\alpha}}\right)$, such that $\mathbf{I}\subseteq M_{1,\alpha+1}$;

equivalently

(e)′ for no $c\in M_2$,

$$c\underset{M_{1,\alpha}}{\not\!\!\bigcup} M_{1,\alpha+1}\quad\text{and}\quad\dim\left(\frac{c}{M_{1,\alpha}},M_2\right)\le\lambda;$$

(f) if $A\subseteq M_{1,\alpha}$, $|A|<\kappa_r(T)+\aleph_1$, $p_i\in S(M_{0,\alpha})$ for $i<i^*<\kappa$; $B\subseteq M_2$, $|B|<\kappa$, $c\in M_2$,

$$\frac{c}{B}\text{ stationary},\quad\dim\left(\frac{c}{B},M_2\right)>\lambda,\quad\frac{c}{B}\not\perp p_i\quad\text{for } i<i^*,$$

then for some elementary mapping h, $\mathrm{Dom}\,h=A\cup B\cup\{c\}$, $h\restriction A=\mathrm{id}$, $h(B\cup\{c\})\subseteq M_{0,\alpha+1}$,

$$\dim\left(\frac{h(c)}{h(B)},M_2\right)>\lambda,\qquad\text{and for } i<i^*\text{ we have }\frac{h(c)}{h(B)}\not\perp p_i.$$

(g) $\mathrm{tp}(M_{0,\alpha}, B^*)$ does not fork over $B^* \cap M_{0,\alpha+1}$ and $A \subseteq M_{0,0}$.

No problem exists in the inductive construction (as T is stable in κ, we have $\kappa = \kappa^{<\kappa_r(T)}$). Let $M_0 = \bigcup_{\alpha<\kappa} M_{0,\alpha}$, $M_1 = \bigcup_{\alpha<\kappa} M_{1,\alpha}$ and $\mathcal{P} = S(M_0)$. By [Sh-a, IV4.14] (or 1.2(1)) there is $\mathbf{J} \subseteq M_1$ independent over M_0 such that: $[p \in S(M_1), p$ does not fork over $M_0 \Rightarrow p \restriction (M_0 \cup \mathbf{J}) \vdash p]$. By 1.2(2) there is $\mathbf{I} \subseteq M_2$ independent over M_0; $\mathbf{J} \subseteq \mathbf{I}$, $\mathbf{I} \backslash \mathbf{J} \subseteq \bigcup_{p \in \mathcal{P}_1} p(M_2)$ where $\mathcal{P}_1 = \{p \in \mathcal{P} : \dim(p, M_2) > \lambda\}$ such that for $p \in \mathcal{P}_1$, $p \restriction_{\mathrm{st}} (M_0 \cup \mathbf{I}) \vdash p \restriction_{\mathrm{st}} M_2$. It suffices to show that

$$p \in \mathcal{P} \Rightarrow p \restriction_{\mathrm{st}} (M_0 \cup \mathbf{I}) \vdash p \restriction_{\mathrm{st}} M_2.$$

By the choice of $\mathbf{J}$ for $\bar{c} \in M_1$, $\dfrac{\bar{c}}{M_0 \cup \mathbf{I}}$ is weakly orthogonal to $p \restriction_{\mathrm{st}} (M_0 \cup \mathbf{J})$ for $p \in \mathcal{P}$, hence also

$$\frac{\bar{c}}{M_0 \cup \mathbf{I}} \perp_{\mathrm{w}} p \restriction_{\mathrm{st}} (M_0 \cup \mathbf{I}) \qquad (\text{for } \bar{c} \in M_1).\dagger$$

Hence (for $p \in \mathcal{P}$) : $p \restriction_{\mathrm{st}} (M_0 \cup \mathbf{I}) \vdash p \restriction_{\mathrm{st}} (M_1 \cup \mathbf{I})$. Let $A \subseteq M_2$ be such that:

(i) $M_1 \cup \mathbf{I} \subseteq A \subseteq M_2$,

(ii) $\bar{c} \in A \Rightarrow \dfrac{\bar{c}}{M_1 \cup \mathbf{I}} \perp_{\mathrm{w}} p \restriction_{\mathrm{st}} (M_1 \cup \mathbf{I})$ for $p \in \mathcal{P}$,

(iii) A is maximal under (i) + (ii).

Easily (by [Sh-a, V3.2]) $A = |M_2'|$, M_2' is $\mathbf{F}^a_\kappa$-saturated (even λ-saturated). If $M_2' = M_2$ we finish, otherwise let $c = c_\kappa \in M_2 \backslash M_2'$, and choose $\mathbf{I} = \{c_\zeta : \zeta < \kappa\} \subseteq M_2'$ indiscernible,

$$\mathrm{Av}(\mathbf{I}, M_2') = \frac{c}{M_2'},$$

and we get a contradiction by 1.1 (only κ there is replaced by $\kappa_r(T) + \aleph_1$ here).

1.4. Theorem. *If T is multi-dimensional, then there is a (non-algebraic, stationary) type orthogonal to the empty set.*

Recall (see [Sh-a, V, Definitions 5.2, 5.3])

1.5. Definition. ($\mathfrak{C}$) A stable theory T is called multi-dimensional if there is $\{\bar{c}^\alpha : \alpha \le \mu\}$ which is multi-dimensional, which means:

(i) $\mu \ge \kappa_r(T)$,

(ii) $\bar{c}^\alpha = \langle c^\alpha_n : n < \omega \rangle$ is an indiscernible set,

(iii) $\{\bar{c}^\alpha : \alpha < \mu\}$ is an indiscernible set,

(iv) letting $\mathbf{I}^\alpha = \{c^\alpha_n : n < \omega\}$, $\{\mathbf{I}^\alpha : \alpha < \mu\} \not\perp_{\mathrm{w}} \mathbf{I}^\mu$, i.e. for some $\mathbf{F}^a_\kappa$-saturated model M, $\bigcup_{\alpha \le \mu} \mathbf{I}^\alpha \subseteq M$, and

$$\dim(\mathbf{I}^\mu, M) < \mathrm{Min}\{\dim(\mathbf{I}^\alpha, M) : \alpha < \mu\}.$$

†By [Sh-a, III4.22].

PROOF OF 1.4. We use 1.5's notation. Let $\kappa = \kappa_r(T) + |T|$. Without loss of generality $\mu > (2^{|T|})^+$; let $\lambda = 2^\mu$, $\lambda_0 = (2^{|T|})^+$; let $\mathbf{J}_\alpha$ $(\alpha \le \mu)$ be such that: $\mathbf{I}^\alpha \cup \mathbf{J}_\alpha$ is an indiscernible set and $\mathbf{J}_\alpha$ is indiscernible over $\bigcup_{\beta\ne\alpha}\mathbf{J}_\beta$ and $|\mathbf{J}_\alpha| = \lambda^+$. Let M_2 be $\mathbf{F}^a_\lambda$-primary over $\bigcup_{\alpha\le\mu}\mathbf{J}_\alpha$, and let $A = \emptyset$. Apply Claim 1.3 (with λ_0 here standing for κ there).

So there are $M_0 \subseteq M_2$ of power λ_0, and $\mathbf{I} \subseteq M_2$ independent over M_0 such that: $|M_0 \cap \mathbf{J}_\alpha| = \lambda_0$ for $\alpha < \lambda_0$,

$$M_0 \underset{M_0 \cap (\bigcup_\alpha \mathbf{J}_\alpha)}{\bigcup\!\!\!\!|\;} \bigcup_\alpha \mathbf{J}_\alpha$$

and for $p \in S(M_0)$, $p\restriction_{\mathrm{st}}(M_0 \cup \mathbf{I}) \vdash p\restriction_{\mathrm{st}} M_2$. By the proof of 1.3 without loss of generality for every $\alpha \le \mu$: either $|\mathbf{J}_\alpha \cap M_0| = \lambda_0$ or $\dfrac{\mathbf{J}_\alpha}{M_0 \cup \bigcup_{\beta\ne\alpha}\mathbf{J}_\beta}$ does not fork over $\bigcup_n(\mathbf{J}_n \cap M_0)$, hence over M_0. By renaming without loss of generality, $|\mathbf{J}_\alpha \cap M_0| = \lambda_0$ iff $\alpha < \lambda_0$. There is M_1, $M_0 \subseteq M_1 \subseteq M_2$, $\|M_1\| = \lambda$, M_1 saturated and $\mathrm{tp}_*(M_1, M_0 \cup \mathbf{I})$ does not fork over $M_0 \cup \mathbf{J}$, where $\mathbf{J} = M_1 \cap \mathbf{I}$ and $|M_1 \cap \mathbf{J}_\alpha| = \lambda$ for $\alpha \le \mu$ and M_2 is $\mathbf{F}^a_\lambda$-constructible over $M_1 \cup \bigcup_{\alpha<\mu}\mathbf{J}_\alpha = M_1 \cup \bigcup_{\alpha<\mu}(\mathbf{J}_\alpha \setminus M_1)$.

Let $M_2' \subseteq M_2$ be $\mathbf{F}^a_\kappa$-primary over $M_1 \cup (\mathbf{I}\setminus\mathbf{J})$. If $M_2 \ne M_2'$, by the conclusion of 1.3 for every $c \in M_2 \setminus M_2'$, $\dfrac{c}{M_2'}$ is (not algebraic and) orthogonal to M_0, hence to $\emptyset$, the desired conclusion.

So assume $M_2 = M_2'$. As any $c \in \mathbf{J}_\mu \setminus M_1$ realizes $\mathrm{Av}(\mathbf{J}_\mu, M_1)$ (and as $M_2 = M_2'$), we have $\mathrm{Av}(\mathbf{J}_\mu, M_1)\ {}_{\mathrm{w}}\!\ge \{\mathrm{tp}(d, M_1) : d \in \mathbf{I}\setminus\mathbf{J}\}$. Now for each $d \in \mathbf{I}\setminus\mathbf{J}$,

$$\frac{d}{M_1}\ {}_{\mathrm{w}}\!\ge \{\mathrm{Av}(\mathbf{J}_\alpha, M_1) : \alpha \le \mu\}$$

(remember M_2 is $\mathbf{F}^a_\lambda$-primary over $M_1 \cup \bigcup_{\alpha\le\mu}(\mathbf{J}_\alpha \setminus M_1)$), hence for some $u_d \subseteq \mu + 1$, $|u_d| < \kappa_r(\tau)$ and

$$\frac{d}{M_1}\ {}_{\mathrm{w}}\!\ge \{\mathrm{Av}(\mathbf{J}_\alpha, M_1) : \alpha \in u_d\}.$$

However, by the choice of $\mathbf{I}$ and M_1, $d \underset{M_0}{\bigcup\!\!\!\!|\;} M_1$, hence (by the choice of M_0) without loss of generality, $u_d \subseteq \lambda_0$; so

$$\frac{d}{M_1}\ {}_{\mathrm{w}}\!\ge \{\mathrm{Av}(\mathbf{J}_\alpha; M_1) : \alpha < \lambda_0\} \qquad (\text{for each } d \in \mathbf{I}\setminus\mathbf{J}).$$

As

$$\mathrm{Av}(\mathbf{J}_\mu, M_1)\ {}_{\mathrm{w}}\!\ge \left\{\frac{d}{M_1} : d \in \mathbf{I}\setminus\mathbf{J}\right\},$$

together (remembering the choice of $\mathbf{J}_\alpha$'s) $\mathrm{Av}(\mathbf{I}^\mu, M_1)\ {}_{\mathrm{w}}\!\geq \{\mathrm{Av}(\mathbf{I}^\alpha, M_1) : \alpha < \mu\}$, a contradiction.

1.6. Conclusion. If T is multi-dimensional, $\kappa_r(T) \leq \aleph_\alpha \leq \aleph_\beta$, T stable in $\aleph_\beta$, *then* T has $\geq 2^{|\beta-\alpha|}$ pairwise non-isomorphic $\mathbf{F}^a_{\aleph_\alpha}$-saturated models of cardinality $\aleph_\beta$.

References

[Sh-a] S. Shelah, *Classification Theory and the Number of Non-isomorphic Models*, North-Holland, Amsterdam, 1978, 542 + xvi pp.

[Sh-c] S. Shelah, *Classification Theory*, revised edition, North-Holland, Amsterdam, 1990, 705 + xxxiv pp.